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LINEAR DIFFEOMORPHISMS WITH LIMIT SHADOWING

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Abstract. In this paper, we show that for a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, f has the limit shadowing property if and only if the matrix A is hyperbolic.

1. Introduction

Let (X, d) be a compact metric space with the metric d, and let $f : X \to X$ be a homeomorphism. For $\delta > 0$, a sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ there is $y \in X$ such that $d(f^n(y), x_n) < \epsilon$ for all $n \in \mathbb{Z}$. We introduce the limit shadowing property which founded in [2]. We say that f has the *limit shadowing property* if there exists $\delta > 0$ with the following property: if a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is δ-limit pseudo orbit of f for which relations $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$, and $d(f^{-1}(x_{i+1}), x_i) \to 0$ as $i \to -\infty$ hold, then there is a point $y \in X$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. It is easy to see that f has the limit shadowing property on Λ if and only if f^n has the limit shadowing property on Λ for $n \in \mathbb{Z} \setminus \{0\}$. Note that the limit shadowing property is not the shadowing property. In fact, in [2], this concept is called the weak limit shadowing property and different from the notion of Pilyugin [3](see, [2] Example 3, 4).

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The notion of the pseudo orbits very often appears in several branches of the modern theory of dynamical system. For instance, the pseudo-orbit tracing property (shadowing property) usually plays an important role in the stability theory(see, [3]).

Let A be a nonsingular matrix on $\mathbb{C}^n$. We consider the dynamical system $f(x) = Ax$ of $\mathbb{C}^n$. We say that the matrix A is called hyperbolic if the spectrum does not intersect the circle $\{\lambda : |\lambda| = 1\}$ (for more detail, see [1]).

THEOREM 1.1. *For a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, the following conditions are mutually equivalent:*

(a) *f has the limit shadowing property,*
(b) *the matrix A is hyperbolic.*

2. Proof of Theorem 1.1

For the proof of $(a) \Rightarrow (b)$, we need the following two lemmas.

LEMMA 2.1. *Let (X, d) be a metric space. Assume that for two dynamical systems f and g on X, there exists a homeomorphism h on X such that $f \circ h = h \circ g$. Then f has the limit shadowing property if and only if g has the limit shadowing property.*

Proof. Suppose that f has the limit shadowing property. For any $\delta > 0$, let $\xi = \{x_i\}_{i \in \mathbb{Z}}$ be a δ-limit pseudo orbit of f. Then $d(f(x_i), x_{i+1}) < \delta$, for all $i \in \mathbb{Z}$ and $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$. Since $f \circ h = h \circ g$, we know that

$$d(g(h^{-1}(x_i)), h^{-1}(x_{i+1})) < \delta \quad \text{for all } i \in \mathbb{Z},$$

and $d(g(h^{-1}(x_i)), h^{-1}(x_{i+1})) \to 0$ as $i \to \pm\infty$. Thus $\{h^{-1}(x_i)\}_{i \in \mathbb{Z}}$ is a δ-limit pseudo orbit of g. Since f has the limit shadowing property, there is a point $y \in X$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Then $d(f^i(y), x_i) = d(g^i(h^{-1}(y)), h^{-1}(x_i)) \to 0$ as $i \to \pm\infty$. Then the point $h^{-1}(y) \in X$ is the limit shadowing point of g. Thus g has the limit shadowing property. □

LEMMA 2.2. [3] *Let A be a nonhyperbolic matrix and λ be an eigenvalue of A with $|\lambda| = 1$. Then there exists a nonsingular matrix T such that $J = T^{-1}AT$ is a Jordan form of A and the matrix J has the form*

$$\begin{pmatrix} B & O \\ O & D \end{pmatrix}$$

where B is the nonsingular $m \times m$ complex matrix with the form

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix},$$

and D is the hyperbolic matrix.

Proof of $(a) \Rightarrow (b)$**.** Suppose that f has the limit shadowing property. To derive a contradiction, we may assume that the matrix A is non-hyperbolic. Then the matrix A has an eigenvalue λ with $|\lambda| = 1$. By Lemma 2.2, there is a nonsingular matrix T such that $J = T^{-1}AT$ is a Jordan form of A and the jordan form $J = \begin{pmatrix} B & O \\ O & D \end{pmatrix}$, where B and D are as in Lemma 2.2. Let $g(x) = J(x) = T^{-1}AT(x)$, and let $h(x) = T(x)$ for $x \in \mathbb{C}^n$. Then $f \circ h = h \circ g$. Since f has the limit shadowing property, by Lemma 2.1, g has the limit shadowing property. Let $\delta > 0$ be the number of the definition of the limit shadowing property of g. Denote by $x^{(i)}$ the i-th component of a vector $x \in \mathbb{C}^n$. Then we construct a δ-limit pseudo orbit as follows:

$$x_{i+1}^{(1)} = \lambda x_i^{(1)}\Big(1 + \frac{\delta}{2^{|i|}|x_i^{(1)}|}\Big),$$

and $x'_{i+1} = (x_{i+1}^{(2)}, x_{i+1}^{(3)}, \ldots, x_{i+1}^{(n)}) = ((Jx_i)^{(2)}, (Jx_i)^{(3)}, \ldots, (Jx_i)^{(n)})$, for all $i \in \mathbb{Z}$. Since $g(x_i) = Jx_i = (\lambda x_i^{(1)}, (Jx_i)^{(2)}, (Jx_i)^{(3)}, \ldots, (Jx_i)^{(n)}) = (\lambda x_i^{(1)}, x'_{i+1})$, we know that if $\lambda = 1$, then

$$d(g(x_i), x_{i+1}) = \Big|x_i^{(1)} - x_i^{(1)} - \frac{x_i^{(1)}\delta}{2^{|i|}|x_i^{(1)}|}\Big| = \frac{\delta}{2^{|i|}} < \delta,$$

for all $i \in \mathbb{Z}$ and if $i \to \pm\infty$, then $d(g(x_i), x_{i+1}) = \delta/2^{|i|} \to 0$. Thus $\{x_i\}_{i\in\mathbb{Z}}$ is a δ-limit pseudo orbit of g. Since g has the limit shadowing property, there is a point $y \in X$ such that $d(g^i(y), x_i) \to 0$ as $i \to \pm\infty$. If $y = (0, 0, \ldots, 0)$ then

$$d(g^{i+1}(y), x_{i+1}) = \Big|x_i^{(1)} + \frac{x_i^{(1)}\delta}{2^{|i|}|x_i^{(1)}|}\Big| \geq |x_i^{(1)}| > 0.$$

This is a contradiction. If $y = (0, y^{(2)}, y^{(3)}, \ldots, y^{(n)})$, then

$$g^{i+1}(y) = (0, (J^i y)^{(2)}, (J^i y)^{(3)}, \ldots, (J^i y)^{(n)}).$$

Then, we see that if for all $i \in \mathbb{Z}$,
$|((Jx_i)^{(2)}, (Jx_i)^{(3)}, \ldots, (Jx_i)^{(n)}) - ((J^i y)^{(2)}, (J^i y)^{(3)}, \ldots, (J^i y)^{(n)})| = 0,$
then as in the proof of the above, for $(J^i y)^{(1)} = 0$, we get a contradiction. Thus we see that for the point $y \in X$, the first component of y, say $y^{(1)}$, is not equal to 0. Then we consider the case $g(y) = g(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) = (y^{(1)}, (Jy)^{(2)}, (Jy)^{(3)}, \ldots, (Jy)^{(n)})$. Thus, for all $i \in \mathbb{Z}$,

$$\Big|x_i^{(1)} + \frac{x_i^{(1)}\delta}{2^{|i|}|x_i^{(1)}|} - y^{(1)}\Big| \geq |x_i^{(1)} - y^{(1)}|.$$

Take $\eta > 0$, let $|x_0^{(1)}| = \eta$. For all $i \in \mathbb{Z}$, we see that

$$|x_i^{(1)}| = \eta + \delta + \frac{\delta}{2} + \frac{\delta}{2^2} + \cdots + \frac{\delta}{2^{i-1}} = \eta + 2\delta\Big(1 - \frac{1}{2^i}\Big). \tag{2.1}$$

If $x_0 = y$ then by (2.1),

$$|x_i^{(1)} - y^{(1)}| \geq |\eta + 2\delta\Big(1 - \frac{1}{2^i}\Big)| - |\eta| \geq |\eta| - |2\delta\Big(1 - \frac{1}{2^i}\Big)| - |\eta|, \tag{2.2}$$

for all $i \in \mathbb{Z}$. Then by (2.2), if $i \to \infty$, then $|x_i^{(1)} - y^{(1)}| \to -|2\delta| \neq 0$. This is a contradiction. Finally, we consider $x_0^{(1)} \neq y^{(1)}$. Since $|x_0^{(1)} - y^{(1)}| \neq 0$, we can take $\gamma > 0$ such that $|x_0^{(1)} - y^{(1)}| = \gamma$. Let $|x_0^{(1)}| = \eta > 0$. Then by (2.2),

$$|x_i^{(1)} - y^{(1)}| \geq |\eta + 2\delta\Big(1 - \frac{1}{2^i}\Big)| - |\eta| - |\gamma| \geq -|2\delta\Big(1 - \frac{1}{2^i}\Big)| - |\gamma|, \tag{2.3}$$

for all $i \in \mathbb{Z}$. Then by (2.3), if $i \to \infty$, then $|x_i^{(1)} - y^{(1)}| \to -|2\delta| - |\gamma| \neq 0$. This is a contradiction. Thus if f has the limit shadowing property, then the matrix A is hyperbolic. □

Finally, we show that $(b) \Rightarrow (a)$, that is proved by Lee [2] as follow.

Lemma 2.3. *Let $f(x) = Ax$ of $\mathbb{C}^n$. If A is the hyperbolic matrix, then f has the limit shadowing property.*

Proof. Denote by E_p the invariant subspace of $T_p\mathbb{C}^n$ corresponding to the eigenvalues λ_i of A such that $|\lambda_i| < 1$, and by F_p the invariant subspace of $T_p\mathbb{C}^n$ corresponding to the eigenvalues λ_i of A such that $|\lambda_i| > 1$. By [3], there exist $C > 0$, $m \in \mathbb{N}$, $0 < \lambda < 1$, and invariant linear subspaces E_p and F_p of $T_p\mathbb{C}^n$ for $p \in \mathbb{C}^n$ such that

(1) $T_p\mathbb{C}^n = E_p \oplus F_p$,
(2) $|A^{mk}(v)| < C\lambda^k|v|,\ v \in E_p,\ k \geq 0,$

(3) $|A^{-mk}(v)| < C\lambda^{-k}|v|, \ v \in F_p, \ k < 0.$

This means that the dynamical system $f^m(x) = A^m(x)$ is hyperbolic. Then by [2], f^m has the limit shadowing property, therefore, f has the limit shadowing property. $\square$

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